Bornologies, selection principles and function spaces

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Abstract

We study some closure-type properties of function spaces endowed with the new topology of strong uniform convergence on a bornology introduced by Beer and Levy in 2009. The study of these function spaces was initiated in [2] and [3]. The properties we study are related to selection principles.

1   Introduction

Our notation and terminology are standard as in [5]. All spaces are assumed to be metric.

If (X, d) is a metric space, $x \in X$, $A \subset X$ and $\varepsilon > 0$ a real number, we write

$$S(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\},$$
$$A^\varepsilon := \bigcup_{a \in A} S(a, \varepsilon),$$

to denote the open ε-ball with center x and the ε-enlargement of A.

Given spaces X and Y we denote by Y^X (resp. $\mathsf{C}(X, Y)$) the set of all functions (resp. all continuous functions) from X into Y. When $Y = \mathbb{R}$ we write $\mathsf{C}(X)$ instead of $\mathsf{C}(X, \mathbb{R})$. $\mathsf{C}_p(X)$ and $\mathsf{C}_k(X)$ denote the set $\mathsf{C}(X)$ endowed with the pointwise topology τ_p and the compact-open topology τ_k, respectively.

Recall that a *bornology* on a metric space (X, d) is a family $\mathfrak{B}$ of nonempty subsets of X which is closed under finite unions, hereditary (i.e. closed under taking nonempty subsets) and forms a cover of X [8], [7]. Throughout the paper we suppose that X does not belong to a bornology $\mathfrak{B}$ on X. A *base* for a bornology $\mathfrak{B}$ on (X, d) is a subfamily $\mathfrak{B}_0$ of $\mathfrak{B}$ which is cofinal in $\mathfrak{B}$ with respect

to the inclusion, i.e. for each $B \in \mathfrak{B}$ there is $B_0 \in \mathfrak{B}_0$ such that $B \subset B_0$. A base is called *closed* (*compact*) if all its members are closed (compact) subsets of X.

For example, if (X, d) is a metric space, then the family $\mathfrak{F}$ of all nonempty finite subsets of X is a bornology on X; it is the smallest bornology on X and has a closed (in fact a compact) base. The largest bornology on X is the set of all nonempty subsets of X. Other important bornologies are: the collection of all nonempty (i) relatively compact subsets (i.e. subsets with compact closure), denoted by $\mathfrak{K}$, (ii) d-bounded subsets, (iii) totally d-bounded subsets.

The following simple facts will be used in the sequel.

Fact 1. If a bornology $\mathfrak{B}$ has a closed base, then $B \in \mathfrak{B}$ implies $\overline{B} \in \mathfrak{B}$.

Fact 2. For every B in a bornology $\mathfrak{B}$ and every $\delta > 0$ it holds $\overline{B^\delta} \subset B^{2\delta}$.

In [2] the notion of strong uniform continuity was introduced: a (not necessarily continuous) mapping $f : X \to Y$ from a metric space (X, d) to a metric space (Y, ρ) is *strongly uniformly continuous* on a subset B of X if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d(x_1, x_2) < \delta$ and $\{x_1, x_2\} \cap B \neq \emptyset$ imply $\rho(f(x_1), f(x_2)) < \varepsilon$. If $\mathfrak{B}$ is a bornology on X, then $f : X \to Y$ is called *strongly uniformly continuous on $\mathfrak{B}$* if f is strongly uniformly continuous on B for each $B \in \mathfrak{B}$.

In the same paper Beer and Levi defined a new topology on the set Y^X of all functions from X into Y, named the topology of strong uniform convergence. They initiated the study of function spaces Y^X and $\mathsf{C}(X, Y)$ with this new topology and characterized metrizability and first countability. Further analysis of these function spaces was done in [3] for several topological properties (for example, (sub)metrizability, first countability, complete metrizabilty, separability, countable tightness, the Fréchet-Urysohn property). We continue the investigation in this area considering properties related to selection principles. Also, we indicate how the idea of strong uniform convergence can be applied to general selection principles theory. For more information about selection principles see the survey papers [10], [20], [21] and references therein.

If $\mathfrak{U}$ and $\mathfrak{V}$ are collections of subsets of a space X. Then:

(1) $\mathsf{S}_1(\mathfrak{U}, \mathfrak{V})$ denotes the selection hypothesis that for each sequence $(U_n : n \in \mathbb{N})$ of elements of $\mathfrak{U}$ there is a sequence $(u_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $u_n \in U_n$ and $\{u_n : n \in \mathbb{N}\}$ is in $\mathfrak{V}$;

(2) $\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V})$ is the selection hypothesis that for each sequence $(U_n : n \in \mathbb{N})$ of elements of $\mathfrak{U}$ there is a sequence $(V_n : n \in \mathbb{N})$ such that V_n is a finite subset of U_n for each $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} V_n \in \mathfrak{V}$.

2 Function spaces

We begin with a definition from [2].

For given metric spaces (X, d) and (Y, ρ) and a bornology $\mathfrak{B}$ **with closed base** on X we denote by $\tau_{\mathfrak{B}}^s$ the *topology of strong uniform convergence on $\mathfrak{B}$* determined by a uniformity on Y^X having as a base the sets of the form

$$[B, \varepsilon]^s := \{(f, g) : \exists \delta > 0 \, \forall x \in B^\delta, \rho(f(x), g(x)) < \varepsilon\} \quad (B \in \mathfrak{B}, \varepsilon > 0).$$

Therefore, for a function $f \in (\mathsf{C}(X, Y), \tau^s_\mathfrak{B})$ the standard local base of f is the collection of sets

$$[B, \varepsilon]^s(f) = \{g \in (\mathsf{C}(X, Y), \tau^s_\mathfrak{B}) : \exists \delta > 0, \, \rho(g(x), f(x)) < \varepsilon, \, \forall x \in B^\delta\} \quad (B \in \mathfrak{B}, \varepsilon > 0).$$

For each bornology $\mathfrak{B}$ with closed base on X the topology $\tau^s_\mathfrak{B}$ on Y^X is finer than the classical topology $\tau_\mathfrak{B}$ of uniform convergence on $\mathfrak{B}$, and if $\mathfrak{B}$ has a compact base, then $\tau^s_\mathfrak{B} = \tau_\mathfrak{B} \leq \tau_k$ on $\mathsf{C}(X, Y)$. In particular, $\tau_p \leq \tau^s_\mathfrak{F} \leq \tau^s_\mathfrak{B} \leq \tau^s_\mathfrak{K} = \tau_k$ on $\mathsf{C}(X)$ (see [2, Corollary 6.6], [3, Remark 3.4]).

The following notion was introduced in [3]. An open cover $\mathcal{U}$ of a metric space (X, d) with a bornology $\mathfrak{B}$ is said to be a *strong $\mathfrak{B}$-cover* of X (or shortly a $\mathfrak{B}^s$-*cover* of X) if $X \notin \mathcal{U}$ and for each $B \in \mathfrak{B}$ there exist $U \in \mathcal{U}$ and $\delta > 0$ such that $B^\delta \subset U$.

In this paper the collection of all strong $\mathfrak{B}$-covers of a space is denoted by $\mathcal{O}_{\mathfrak{B}^s}$. We also suppose that all spaces we consider are $\mathfrak{B}^s$-*Lindelöf*, i.e. each $\mathfrak{B}^s$-cover contains a countable $\mathfrak{B}^s$-subcover.

Let us define also the following. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be a γ_s-*cover* ($\gamma_{\mathfrak{B}^s}$-*cover*, called a $\mathfrak{B}^s$-*sequence* in [3]) if it is infinite and for $x \in X$ (each $B \in \mathfrak{B}$) there are $n_0 \in \mathbb{N}$ and a sequence $(\delta_n : n \geq n_0)$ of positive real numbers such that $S(x, \delta_n) \subset U_n$ $(B^{\delta_n} \subset U_n)$ for all $n \geq n_0$.

For a given bornology $\mathfrak{B}$ in a space X we denote by Γ_s and $\Gamma_{\mathfrak{B}^s}$ the collection of all countable γ_s-covers and the collection of all $\gamma_{\mathfrak{B}^s}$-covers of X.

The symbol $\underline{0}$ denotes the constantly zero function in $(\mathsf{C}(X), \tau^s_\mathfrak{B})$. The space $(\mathsf{C}(X), \tau^s_\mathfrak{B})$ is homogeneous so that it suffices to look at the point $\underline{0}$ when studying local properties of this space.

In [3] the following theorem was proved.

Theorem 2.1 *Let (X, d) be a metric space and $\mathfrak{B}$ a bornology on X with closed base. The following are equivalent:*

(1) $(\mathsf{C}(X), \tau^s_\mathfrak{B})$ *has countable tightness;*

(2) X *is a $\mathfrak{B}^s$-Lindelöf space.*

Our aim is to prove similar results for countable fan tightness and countable strong fan tightness.

For a space X and a point $x \in X$ the symbol Ω_x denotes the set $\{A \subset X \setminus \{x\} : x \in \overline{A}\}$, and Σ_x is the set of sequences converging to x.

A space X has *countable fan tightness* [1] if for each $x \in X$ we have that $\mathsf{S}_{fin}(\Omega_x, \Omega_x)$ holds. X has *countable strong fan tightness* [19] if for each $x \in X$ the selection principle $\mathsf{S}_1(\Omega_x, \Omega_x)$ holds.

Theorem 2.2 *Let (X, d) be a metric space and $\mathfrak{B}$ a bornology on X with closed base. The following are equivalent:*

(1) $(\mathsf{C}(X), \tau^s_\mathfrak{B})$ *has countable strong fan tightness;*

(2) X *satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.*

Proof. $(1) \Rightarrow (2)$: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open strong $\mathfrak{B}$-covers of X. For every $n \in \mathbb{N}$ and every $B \in \mathfrak{B}$ there exists $U_B \in \mathcal{U}_n$ and $\delta > 0$ such that $B^{2\delta} \subset U_B$.

For a fixed $n \in \mathbb{N}$ and $B \in \mathfrak{B}$ let

$$\mathcal{U}_{n,B} := \{U \in \mathcal{U}_n : B^{2\delta} \subset U\}.$$

For each $U \in \mathcal{U}_{n,B}$, by Fact 2, there is a continuous function $f_{B,U}$ from X into $[0,1]$ such that $f_{B,U}(B^\delta) = \{0\}$ and $f_{B,U}(X \setminus U) = \{1\}$. Let for each n,

$$A_n = \{f_{B,U} : B \in \mathfrak{B}, U \in \mathcal{U}_{n,B}\}.$$

It is easily seen that $\underline{0}$ belongs to the closure of A_n in $\tau_{\mathfrak{B}}^s$ for each $n \in \mathbb{N}$. By (1) there is a sequence $(f_{B_n,U_n} : n \in \mathbb{N})$ such that for each n, $f_{B_n,U_n} \in A_n$ and $\underline{0} \in \overline{\{f_{B_n,U_n} : n \in \mathbb{N}\}}$. We claim that the sequence $(U_n : n \in \mathbb{N})$ witnesses that X has property $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Let $B \in \mathfrak{B}$. Since $\underline{0} \in \overline{\{f_{B_n,U_n} : n \in \mathbb{N}\}}$ it follows that there is $m \in \mathbb{N}$ such that $[B,1]^s(\underline{0})$ contains the function f_{B_m,U_m}. Therefore, there is $\delta > 0$ such that for each $x \in B^\delta$ it holds $f_{B_m,U_m}(x) < 1$, which means $B^\delta \subset U_m$.

$(2) \Rightarrow (1)$: Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $(\mathsf{C}(X), \tau_{\mathfrak{B}}^s)$ whose closures contain $\underline{0}$.

For every $B \in \mathfrak{B}$ and every $m \in \mathbb{N}$ the neighborhood $[B, 1/m]^s(\underline{0})$ of $\underline{0}$ intersects each A_n. It follows that for each $n \in \mathbb{N}$ there exists a function $f_{B,n,m} \in A_n$ satisfying: there is $\delta > 0$ with $|f_{B,n,m}(x)| < 1/m$ for each $x \in B^\delta$. For each n set

$$\mathcal{U}_{n,m} = \{f^\leftarrow(-1/m, 1/m) : m \in \mathbb{N}, f \in A_n\}.$$

(We can view the indices m, n in $\mathcal{U}_{n,m}$ as $\varphi(m,n)$ for some bijection $\varphi : \mathbb{N}^2 \to \mathbb{N}$.) We claim that for each $n, m \in \mathbb{N}$, $\mathcal{U}_{n,m}$ is a $\mathfrak{B}^s$-cover of X. Indeed, if $B \in \mathfrak{B}$, then there is $f_{B,n,m} \in [B, 1/m]^s(\underline{0}) \cap A_n$. Hence there is $\delta > 0$ such that $|f_{B,n,m}(x)| < 1/m$ for each $x \in B^\delta$. This means $B^\delta \subset f_{B,n,m}^\leftarrow(-1/m, 1/m) \in \mathcal{U}_{n,m}$.

Put $M = \{m \in \mathbb{N} : X \in \mathcal{U}_{n,m} \text{ for some } n \in \mathbb{N}\}$.

Case 1. M is infinite.

There are $m_1 < m_2 < \cdots$ in M and (the corresponding) $n_1, n_2, \cdots$ in $\mathbb{N}$ such that $f_{B_i,n_i,m_i}^\leftarrow(-1/m_i, 1/m_i) = X$ for all $i \in \mathbb{N}$ and some $B_i \in \mathfrak{B}$. Let $[B, \varepsilon]^s(\underline{0})$ be a $\tau_{\mathfrak{B}}^s$-neighbourhood of $\underline{0}$. Pick m_k such that $1/m_k < \varepsilon$. For every $m_i > m_k$ we have $f_{B_i,n_i,m_i}(x) \in (-1/m_i, 1/m_i)$ for each $x \in X$ and so $f_{B_i,n_i,m_i} \in [B, 1/m_i]^s(\underline{0}) \subset [B, \varepsilon]^s(\underline{0})$. This means that the sequence $(f_{B_i,n_i,m_i} : i \in \mathbb{N})$ converges to $\underline{0}$, hence $\underline{0} \in \overline{\{f_{B_i,n_i,m_i} : i \in \mathbb{N}\}}$.

Case 2. M is finite.

There is $m_0 \in \mathbb{N}$ such that for each $m \geq m_0$ and each $n \in \mathbb{N}$, the set $\mathcal{U}_{n,m}$ is a $\mathfrak{B}^s$-cover of X. One may suppose $m_0 = 1$, and since the set $\{n \in \mathbb{N} : X \in \mathcal{U}_{n,n}\}$ is also finite we can work with this set instead of M, and assume that for each

$n \in \mathbb{N}, \mathcal{U}_{n,n}$ is a $\mathfrak{B}^s$-cover of X. By (2) choose for each $n \in \mathbb{N}$ a set $U_{n,n} \in \mathcal{U}_{n,n}$ so that $\{U_{n,n} : n \in \mathbb{N}\}$ is a $\mathfrak{B}^s$-cover of X. Consider the corresponding functions $f_{B_n,n,n}$, $n \in \mathbb{N}$. We claim that $\underline{0} \in \overline{\{f_{B_n,n,n} : n \in \mathbb{N}\}}$. Let $[B,\varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$. There are $j \in \mathbb{N}$ and $\delta > 0$ such that $B^\delta \subset U_{j,j}$. But the set K of all such j is infinite because $\{U_{n,n} : n \in \mathbb{N}\}$ is a $\mathfrak{B}^s$-cover of X. Take $k \in K$ so that $1/k < \varepsilon$. Then $B^\delta \subset f^{\leftarrow}_{B_k,k,k}(-1/k, 1/k) \subset f^{\leftarrow}_{B_k,k,k}(-\varepsilon, \varepsilon)$, i.e. $f_{B_k,k,k} \in [B,\varepsilon]^s(\underline{0})$. ▲

Let $\mathcal{K}$ denote the family of k-covers of a space X. (An open cover is a k-cover if each compact set $K \subset X$ is contained in a member of the cover.)

Corollary 2.3 *(([9]) The space $\mathsf{C}_k(X)$ has countable strong fan tightness if and only if X has property $\mathsf{S}_1(\mathcal{K},\mathcal{K})$.*

The following theorem can be proved similarly.

Theorem 2.4 *Let (X,d) be a metric space and $\mathfrak{B}$ a bornology with closed base on X. The following are equivalent:*

(1) *$(\mathsf{C}(X), \tau^s_{\mathfrak{B}})$ has countable fan tightness;*

(2) *X satisfies $\mathsf{S}_{fin}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.*

Corollary 2.5 *(([14], [9]) The space $\mathsf{C}_k(X)$ has countable fan tightness if and only if X has property $\mathsf{S}_{fin}(\mathcal{K},\mathcal{K})$.*

Recall that a space X is said to be *Fréchet-Urysohn* if for each A subset of X and each $x \in \overline{A}$ there is a sequence in A converging to x. X is *strictly Fréchet-Urysohn* if fulfills the selection property $\mathsf{S}_1(\Omega_x, \Sigma_x)$.

Theorem 2.6 *Let (X,d) be a metric space and $\mathfrak{B}$ be a bornology on X. The following are equivalent:*

(1) *$(\mathsf{C}(X), \tau^s_{\mathfrak{B}})$ is a strictly Fréchet-Urysohn space;*

(2) *X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

Proof. $(1) \Rightarrow (2)$: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\mathfrak{B}^s$-covers of X. For every $n \in \mathbb{N}$ and every $B \in \mathfrak{B}$ there exist $U_{B,n} \in \mathcal{U}_n$ and $\delta > 0$ such that $B^{2\delta} \subset U_{B,n}$. Set $\mathcal{U}_{n,B} := \{U \in \mathcal{U}_n : B^{2\delta} \subset U\}$. For each $U \in \mathcal{U}_{n,B}$ pick a continuous function $f_{B,U}$ from X into $[0,1]$ such that $f_{B,U}(B^\delta) = \{0\}$ and $f_{B,U}(X \setminus U) = \{1\}$. Let for each n,

$$A_n = \{f_{B,U} : B \in \mathfrak{B}, U \in \mathcal{U}_{n,B}\}.$$

Clearly the function $\underline{0}$ belongs to the $\tau^s_{\mathfrak{B}}$-closure of A_n for each $n \in \mathbb{N}$, and since $(\mathsf{C}(X), \tau^s_{\mathfrak{B}})$ is strictly Fréchet-Urysohn, there are $f_{B_n, U_n} \in A_n$, $n \in \mathbb{N}$, such that the sequence $(f_{B_n, U_n} : n \in \mathbb{N})$ $\tau^s_{\mathfrak{B}}$-converges to $\underline{0}$. We prove that the set $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. For the neighbourhood $[B,1]^s(\underline{0})$ of $\underline{0}$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ $f_{B_n, U_n} \in [B,1]^s(\underline{0})$. So, for each $n > n_0$ there is $\delta_n > 0$ such that $B^{\delta_n} \subset f^{\leftarrow}_{B_n, U_n}(-1,1)$, hence $B^{\delta_n} \subset U_n$.

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(2) ⇒ (1): Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $(\mathsf{C}(X), \tau_{\mathfrak{B}}^s)$ such that $\underline{0} \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$. For every $B \in \mathfrak{B}$ and every $m \in \mathbb{N}$ the neighborhood $[B, 1/m]^s(\underline{0})$ of $\underline{0}$ intersects each A_n, and thus for each $n \in \mathbb{N}$ there is a function $f_{B,n,m} \in A_n$ such that there is $\delta > 0$ with $|f_{B,n,m}(x)| < 1/m$ for each $x \in B^\delta$. For each $n \in \mathbb{N}$ let

$$\mathcal{U}_{n,m} = \{f^{\leftarrow}(-1/m, 1/m) : m \in \mathbb{N}, f \in A_n\}.$$

As in the proof of Theorem 2.2 we conclude that for each $n, m \in \mathbb{N}$, $\mathcal{U}_{n,m}$ is a $\mathfrak{B}^s$-cover of X. Apply now assumption (2) to the sequence $(\mathcal{U}_{n,n} : n \in \mathbb{N})$ and for each n pick an element $U_{n,n}$ in $\mathcal{U}_{n,n}$ such that the set $\{U_{n,n} : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. To each $U_{n,n}$ associate the corresponding function $f_{B_n,n,n} \in A_n$. We prove that the sequence $(f_{B_n,n,n} : n \in \mathbb{N})$ converges to $\underline{0}$.

Let $[B, \varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$. Since $\{U_{n,n} : n \in \mathbb{N}\} \in \Gamma_{\mathfrak{B}^s}$, there are $m \in \mathbb{N}$ and $\delta_n > 0$, $n > m$, such that $1/m < \varepsilon$ and for each $n > m$, $B^{2\delta_n} \subset U_{n,n}$. Therefore, for all $n > m$, $f_{B_n,n,n}(B^{\delta_n}) \subset (-1/n, 1/n) \subset (-\varepsilon, , \varepsilon)$, i.e. $f_{B_n,n,n} \in [B, \varepsilon]^s(\underline{0})$. ▲

Theorem 2.7 *If (X, d) is a metric space and $\mathfrak{B}$ a bornology with closed base on X, then the following statements are equivalent:*

(1) *X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$;*

(2) *Each $\mathfrak{B}^s$-cover $\mathcal{U}$ of X contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a $\gamma_{\mathfrak{B}^s}$-cover of X.*

Proof. Obviously (1) ⇒ (2) and thus we prove (2) ⇒ (1).

Proof. (2) ⇒ (1): [Observe first that each space satisfying (2) is $\mathfrak{B}^s$-Lindelöf.] Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $\mathfrak{B}^s$-covers of X. Construct a new sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of $\mathfrak{B}^s$-covers of X in the following way:

(i) $\mathcal{V}_1 = \mathcal{U}_1$;

(ii) $\mathcal{V}_{n+1}$ is a refinement of $\mathcal{V}_n$ and $\mathcal{U}_n$.

As the bornology $\mathfrak{B}$ has closed base, by Fact 1, for each $B \in \mathfrak{B}$ the closure of B is also in $\mathfrak{B}$. On the other hand, $X \notin \mathfrak{B}$, so that for every $B \in \mathfrak{B}$ there is a point $x_B \in X \setminus \overline{B}$. It follows there is $\delta > 0$ such that $x_B \notin B^\delta$, i.e. $B^\delta \subset X \setminus \{x_B\}$. Therefore, $\{X \setminus \{x\} : x \in X\}$ is a $\mathfrak{B}^s$-cover of X. Applying assumption (2) to $\{X \setminus \{x\} : x \in X\}$ we find a sequence $(x_n : n \in \mathbb{N})$ of points of X such that $\{X \setminus \{x_n\} : n \in \mathbb{N}\}$ is still a $\mathfrak{B}^s$-cover of X. For each $n \in \mathbb{N}$ denote

$$\mathcal{W}_n = \{V \setminus \{x_n\} : V \in \mathcal{V}_n\} \text{ and } \mathcal{W} = \cup\{\mathcal{W}_n : n \in \mathbb{N}\}.$$

We claim that $\mathcal{W}$ is a $\mathfrak{B}^s$-cover of X.

Let $B \in \mathfrak{B}$. There are $\delta > 0$ and $k \in \mathbb{N}$ such that $B^\delta \subset X \setminus \{x_k\}$. Also, since $\mathcal{V}_k$ is a $\mathfrak{B}^s$-cover, there are $\mu > 0$ and $V \in \mathcal{V}_k$ such that $B^\mu \subset V$. Then for $\varepsilon = \min\{\delta, \mu\}$ we have $B^\varepsilon \subset V \setminus \{x_k\} \in \mathcal{W}_k \subset \mathcal{W}$.

By (2), there exists a sequence $(W_m : m \in \mathbb{N})$ in $\mathcal{W}$ so that $\{W_m : m \in \mathbb{N}\} \in \Gamma_{\mathfrak{B}^s}$. For every $m \in \mathbb{N}$ there exist $n_m \in \mathbb{N}$ and a set $V_{n_m} \in \mathcal{V}_{n_m}$ with $W_m \subset V_{n_m}$. Let for each $m \in \mathbb{N}$, $F_m = \{x_1, ..., x_m\}$; clearly, $F_m \in \mathfrak{B}$.

Choose $W_{m_1} \in \{W_m : m \in \mathbb{N}\}$ such that $F_1^\delta \subset W_{m_1}$ for some $\delta > 0$; note that $m_1 > 1$. For each $p > 1$ pick $W_{m_p} \in \{W_m : m \in \mathbb{N}\}$ so that $F_p^\mu \subset W_{m_p}$ for some $\mu > 0$, $m_p > m_{p-1}$ and $m_p > p$. Such a choice is possible for each $p \in \mathbb{N}$ because the set $\{W_m : m \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Since $\mathcal{V}_{n+1}$ is a refinement of both $\mathcal{V}_n$ and $\mathcal{U}_n$, for each $i \in \mathbb{N}$ with $n_{m_p} < i \leq n_{m_{p+1}}$ pick $U_i \in \mathcal{U}_i$ such that $U_i \subset U_j$ for $i > j$. Put $n_{m_0} = 0$ and for each $n \in \mathbb{N}$ define (i) $O_n = U_{n_{m_p}}$ if $n = n_{m_p}$, and (ii) $O_n = U_n$ if $n_{m_p} < n < n_{m_{p+1}}$.

We claim that the sequence $(O_n : n \in \mathbb{N})$ is a selector for $(\mathcal{U}_n : n \in \mathbb{N})$ witnessing that X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. Indeed, let $B \in \mathfrak{B}$. Then there exist $m_0 \in \mathbb{N}$ and a sequence $(\delta_n : n > m_0)$ such that for every $m > m_0$, $B^{\delta_n} \subset U_m$. By construction of the sets O_n, for all $n > n_{m_0}$ we have $B^{\delta_n} \subset O_n$. ▲

In [3] it was proved:

Theorem 2.8 ([3]) *If (X, d) is a metric space and $\mathfrak{B}$ a bornology with closed base on X, then the following are equivalent:*

(1) *$(\mathsf{C}(X), \tau_{\mathfrak{B}}^s)$ is a Fréchet-Urysohn space;*

(2) *Each $\mathfrak{B}^s$-cover $\mathcal{U}$ of X contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a $\gamma_{\mathfrak{B}^s}$-cover of X.*

From this theorem together with Theorem 2.6 and Theorem 2.7 one obtains the following corollary. (Similar well-known results were obtained independently in [6] and [18] for the space $\mathsf{C}_p(X)$ (see [1]). Compare also with [4], [14], [15] in connection with the space $\mathsf{C}_k(X)$.)

Corollary 2.9 *If (X, d) is a metric space and $\mathfrak{B}$ a bornology with closed base on X, then the following assertions are equivalent:*

(1) *$(\mathsf{C}(X), \tau_{\mathfrak{B}}^s)$ is a Fréchet-Urysohn space;*

(2) *$(\mathsf{C}(X), \tau_{\mathfrak{B}}^s)$ is a strictly Fréchet-Urysohn space;*

(3) *Each $\mathfrak{B}^s$-cover $\mathcal{U}$ of X contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a $\gamma_{\mathfrak{B}^s}$-cover of X;*

(4) *X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

We close this section considering a property similar to the strict Fréchet-Urysohn property.

A space X is called a *selectively strictly A-space* (shortly SSA) [17] if for each sequence $(A_n : n \in \mathbb{N})$ of subsets of X and each point $x \in X$ such that $x \in \overline{A_n} \setminus A_n$ for each $n \in \mathbb{N}$, there is a sequence $(T_n : n \in \mathbb{N})$ such that for each n $T_n \subset A_n$, and $x \in \overline{\bigcup_{n \in \mathbb{N}} T_n} \setminus \bigcup_{n \in \mathbb{N}} \overline{T_n}$.

Theorem 2.10 *Let (X, d) be a metric space and $\mathfrak{B}$ a bornology with closed base on X. Then the following are equivalent:*

(1) *$(\mathsf{C}(X), \tau_{\mathfrak{B}}^s)$ is SAA;*

(2) *for each sequence* $(\mathcal{U}_n : n \in \mathbb{N})$ *of* $\mathfrak{B}^s$*-covers of* X *there is a sequence* $(\mathcal{V}_n : n \in \mathbb{N})$ *such that* $\mathcal{V}_n \subset \mathcal{U}_n$ *for each* n, *no* $\mathcal{V}_n$ *is a* $\mathfrak{B}^s$*-cover of* X, *and* $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ *is a* $\mathfrak{B}^s$*-cover of* X.

Proof. $(1) \Rightarrow (2)$: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\mathfrak{B}^s$-covers of X. For every $n \in \mathbb{N}$ and every $B \in \mathfrak{B}$ there are $U_B \in \mathcal{U}_n$ and $\delta > 0$ with $B^{2\delta} \subset U_B$.

For $n \in \mathbb{N}$, $B \in \mathfrak{B}$ put

$$\mathcal{U}_{n,B} := \{U \in \mathcal{U}_n : B^{2\delta} \subset U\}.$$

For each $U \in \mathcal{U}_{n,B}$ pick a continuous function $f_{n,B,U} : X \to [0,1]$ satisfying $f_{n,B,U}(B^\delta) = \{0\}$ and $f_{n,B,U}(X \setminus U) = \{1\}$ and denote

$$A_n = \{f_{n,B,U} : B \in \mathfrak{B}, U \in \mathcal{U}_{n,B}\} \ (n \in \mathbb{N}).$$

The function $\underline{0}$ belongs to the $\tau_\mathfrak{B}^s$-closure of A_n as it is easy to verify. On the other hand, $\underline{0} \notin A_n$ for each $n \in \mathbb{N}$. Otherwise $\underline{0} = f_{m,B,U} \in A_m$ for some m, thus $f_{m,B,U}(X \setminus U) = \{0\}$, a contradiction.

By (1), there is a sequence $(T_n : n \in \mathbb{N})$ such that for each n $\quad T_n \subset A_n$ and $\underline{0} \in \overline{\bigcup_{n \in \mathbb{N}} T_n} \setminus \bigcup_{n \in \mathbb{N}} \overline{T_n}$. Denote by $\mathcal{V}_n$, $n \in \mathbb{N}$, the set of corresponding sets U for each $f_{n,B,U} \in T_n$. No $\mathcal{V}_n$ is a $\mathfrak{B}^s$-cover of X, because otherwise $\underline{0} \in \overline{T_n}$. It remains to prove $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n \in \mathcal{O}_{\mathfrak{B}^s}$. Let $B \in \mathfrak{B}$. From $\underline{0} \in \overline{\bigcup_{n \in \mathbb{N}} T_n}$ it follows the existence of $m \in \mathbb{N}$ and $f_{m,B,U} \in T_m$ such that there is $\delta > 0$ with $f_{m,B,U}(x) = 0$ for each $x \in B^\delta$. This means $B^{2\delta} \subset U \in \mathcal{V}_m \subset \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$.

$(2) \Rightarrow (1)$: Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $(\mathsf{C}(X), \tau_\mathfrak{B}^s)$ such that $\underline{0} \in \overline{A_n} \setminus A_n$, $n \in \mathbb{N}$. We proceed as in the proofs of $(2) \Rightarrow (1)$ of Theorems 2.2 and 2.6. For each $B \in \mathfrak{B}$ and each $m \in \mathbb{N}$ the neighborhood $[B, 1/m]^s(\underline{0})$ of $\underline{0}$ intersects each A_n. Therefore, for each $n \in \mathbb{N}$ there is a function $f_{B,n,m} \in A_n$ such that there is $\delta > 0$ with $|f_{B,n,m}(x)| < 1/m$ for each $x \in B^\delta$. For each $n, m \in \mathbb{N}$ define

$$\mathcal{U}_{n,m} = \{f^\leftarrow(-1/m, 1/m) : m \in \mathbb{N}, f \in A_n\}.$$

All $\mathcal{U}_{n,m}$ are $\mathfrak{B}^s$-covers of X.

Case 1: $X \in \mathcal{U}_{n,n}$ for infinitely many n.

Then there exist an increasing sequence $n_1 < n_2 < \ldots < n_k < \ldots$ in $\mathbb{N}$ and $f_{n_k} \in A_{n_k}$, $k \in \mathbb{N}$, such that $f_{n_k}^\leftarrow(-1/n_k, 1/n_k) = X$. Put $T_{n_k} = \{f_{n_k}\}$, $k \in \mathbb{N}$, and $T_n = \emptyset$ for $n \neq n_k$, $k \in \mathbb{N}$. It is easy to see that $\underline{0} \in \overline{\bigcup_{n \in \mathbb{N}} T_n} \setminus \bigcup_{n \in \mathbb{N}} \overline{T_n}$ (because the sequence $(f_{n_k} : k \in \mathbb{N})$ actually $\tau_\mathfrak{B}^s$-converges to $\underline{0}$).

Case 2: $X \in \mathcal{U}_{n,n}$ for finitely many n.

Suppose that $X \notin \mathcal{U}_{n,n}$ for each n. Choose a sequence $(\mathcal{V}_{n,n} : n \in \mathbb{N})$ witnessing (2). For each $V \in \mathcal{V}_{n,n}$ pick a function $f_V \in A_n$ with $V = f_V^\leftarrow(-1/n, 1/n)$ and put $T_n = \{f_V : V \in \mathcal{V}_{n,n}\}$. Let us show that $(T_n : n \in \mathbb{N})$ is as required in (1).

First, $\underline{0} \notin \bigcup_{n \in \mathbb{N}} \overline{T_n}$. Otherwise, $\underline{0} \in \overline{T_m}$ for some m would imply that $\mathcal{V}_{m,m}$ is a $\mathfrak{B}^s$-cover of X.

Second, we prove $\underline{0} \in \overline{\bigcup_{n \in \mathbb{N}} T_n}$. Let $[B, \varepsilon]^s(\underline{0})$, $B \in \mathfrak{B}$, $\varepsilon > 0$, be a neighborhood of $\underline{0}$. Since $\bigcup_{n \in \mathbb{N}} \mathcal{V}_{n,n}$ is a $\mathfrak{B}^s$-cover of X there are $\delta > 0$ and a natural number m such that $1/m < \varepsilon$ and for some $V \in \mathcal{V}_{m,m}$ we have $B^\delta \subset V = f_V^{\leftarrow}(-1/m, 1/m)$. Hence $f_V \in T_m$ and since obviously $f_V \in [B, \varepsilon]^s(\underline{0})$, we conclude $\underline{0} \in \overline{\bigcup_{n \in \mathbb{N}} T_n}$. ▲

3   Covering properties

In this short section we indicate how the idea of strong uniform convergence on a bornology may be further applied to selection principles theory.

According to [12], we say that a space X with a bornology $\mathfrak{B}$ satisfies the selection principle $\alpha_i(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, $i = 2, 3, 4$, if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $\gamma_{\mathfrak{B}^s}$-covers of X there is a $\gamma_{\mathfrak{B}^s}$-cover $\mathcal{V}$ of X such that:

$\alpha_2(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$: for each $n \in \mathbb{N}$, $\mathcal{U}_n \cap \mathcal{V}$ is infinite;

$\alpha_3(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$: for infinitely many $n \in \mathbb{N}$, $\mathcal{U}_n \cap \mathcal{V}$ is infinite;

$\alpha_4(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$: for infinitely many $n \in \mathbb{N}$, $\mathcal{U}_n \cap \mathcal{V} \neq \emptyset$.

Theorem 3.1 *For a metric space (X, d) and a bornology $\mathfrak{B}$ on X the following are equivalent:*

(1) *X satisfies $\alpha_2(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$;*

(2) *X satisfies $\alpha_3(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$;*

(3) *X satisfies $\alpha_4(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$;*

(4) *X satisfies $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

Proof. Clearly $(1) \Rightarrow (2) \Rightarrow (3)$.

$(3) \Rightarrow (4)$: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\gamma_{\mathfrak{B}^s}$-covers of X. Suppose that $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$, $n \in \mathbb{N}$. For all $n, m \in \mathbb{N}$ define $V_{n,m} = U_{1,m} \cap U_{2,m} \cap \cdots \cap U_{n,m}$. Then for each n the set $\mathcal{V}_n = \{V_{n,m} : m \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. (Indeed, fix n and take a $B \in \mathfrak{B}$. Since each $\mathcal{U}_i$, $i \leq n$, is a $\gamma_{\mathfrak{B}^s}$-cover of X, there are m_i and sequences $(\delta_m^{(i)} : m \geq m_i)$, $i \leq n$, of positive real numbers, such that $B^{\delta_m^{(i)}} \subset U_{i,m}$ for every $m \geq m_i$. The sequence $(\delta_m^{(i)} : i \leq n, m \geq m_0)$ and $m_0 = \max\{m_i : i \leq n\}$ witness that $\mathcal{V}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover.) By (3) (and the fact that an infinite subset of a $\gamma_{\mathfrak{B}^s}$-cover is also such a cover) there is an increasing sequence $n_0 = 1 \leq n_1 < n_2 < \cdots$ in $\mathbb{N}$ and a $\gamma_{\mathfrak{B}^s}$-cover $\mathcal{V} = \{V_{n_i,m_i} : i \in \mathbb{N}\}$ such that for each $i \in \mathbb{N}$, $V_{n_i,m_i} \in \mathcal{V}_{n_i}$. For each $i \geq 0$, each j with $n_i < j \leq n_{i+1}$ and each $V_{n_{i+1},m_{i+1}} = U_{1,m_{i+1}} \cap \cdots \cap U_{n_{i+1},m_{i+1}}$ let U_j be the set $U_{j,m_{i+1}}$. The sequence $(U_n : n \in \mathbb{N})$ testifies that X satisfies $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

$(4) \Rightarrow (1)$: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\gamma_{\mathfrak{B}^s}$-covers of X. Suppose as above $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$, $n \in \mathbb{N}$. As we mentioned before every infinite subset of a $\gamma_{\mathfrak{B}^s}$-cover is itself a $\gamma_{\mathfrak{B}^s}$-cover. Thus from each $\mathcal{U}_n$ we can built

countably many disjoint $\gamma_{\mathfrak{B}^s}$-covers $\mathcal{U}_n^k$, $k \in \mathbb{N}$. Apply now (4) to the sequence $(\mathcal{U}_n^k : n, k \in \mathbb{N})$. One obtains a $\gamma_{\mathfrak{B}^s}$-cover $\{U_n^k : U_n^k \in \mathcal{U}_n^k; n, k \in \mathbb{N}\}$ which contains infinitely many elements from each $\mathcal{U}_n$. ▲

Observe that similar results are true for the pairs (Γ_s, Γ_s), $(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_s)$ and $(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

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